Solution to the F-F Polarizatⁿ Problem

$e^{-i\mathcal{H}^{FuF}t}$ can be calculated with aid of

1) $\exp -i\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{bmatrix} t = e^{-i/2\, t}\, e^{-it\begin{bmatrix} -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 \end{bmatrix}}$, $\cos\varphi = 1/\sqrt{3}$, $\sin\varphi = -\sqrt{2/3}$

2) $= e^{-i/2\, t}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cos\frac{\sqrt{3}}{2}t + i\begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}\sin\frac{\sqrt{3}}{2}t\right)$

$\therefore e^{-i\mathcal{H}^{FuF}t}$ has the following twelve terms

(note, all t's to be interpreted as $\omega_0 t$)

3a) $\equiv e^{it}\Big(\underbrace{|\!\uparrow\rangle\langle\uparrow\!|}_{\text{muon spin}}\ \underbrace{|1,1\rangle\langle 1,1|}_{\text{F+F spin 1}} + |\!\downarrow\rangle\langle\downarrow\!|\ |1,-1\rangle\langle 1,-1|\Big)$

$+\left(|\!\uparrow\rangle\langle\uparrow\!| + |\!\downarrow\rangle\langle\downarrow\!|\right)\underbrace{|0\rangle\langle 0|}_{\text{F+F spin }\emptyset}$

$+ e^{-it/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{i}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)\left(|\!\uparrow\rangle\langle\uparrow\!| + |\!\downarrow\rangle\langle\downarrow\!|\right)|1,0\rangle\langle 1,0|$

$+ e^{-it/2}\left(\cos\frac{\sqrt{3}}{2}t - \frac{i}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)\Big[|\!\uparrow\rangle\langle\uparrow\!|\ |1,-1\rangle\langle 1,-1| + |\!\downarrow\rangle\langle\downarrow\!|\ |1,1\rangle\langle 1,1|\Big]$

$-i\sqrt{\frac{2}{3}}\, e^{-it/2}\sin\frac{\sqrt{3}}{2}t\Big[|\!\uparrow\rangle\langle\downarrow\!|\left(|1,0\rangle\langle 1,1| + |1,-1\rangle\langle 1,0|\right) + |\!\downarrow\rangle\langle\uparrow\!|\left(|1,1\rangle\langle 1,0| + |1,0\rangle\langle 1,-1|\right)\Big]$

3b) $= \left(|\!\uparrow\rangle\langle\uparrow\!| + |\!\downarrow\rangle\langle\downarrow\!|\right)\Big[|0\rangle\langle 0| + e^{-it/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{i}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)|1,0\rangle\langle 1,0|\Big]$

$+\left(e^{it}\,|\!\uparrow\rangle\langle\uparrow\!| + e^{-it/2}\left(\cos\frac{\sqrt{3}}{2}t - \frac{i}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)|\!\downarrow\rangle\langle\downarrow\!|\right)|1,1\rangle\langle 1,1|$

$+\left(e^{it}\,|\!\downarrow\rangle\langle\downarrow\!| + e^{-it/2}\left(\cos\frac{\sqrt{3}}{2}t - \frac{i}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)|\!\uparrow\rangle\langle\uparrow\!|\right)|1,-1\rangle\langle 1,-1|$

$-i\sqrt{\frac{2}{3}}\, e^{-it/2}\sin\frac{\sqrt{3}}{2}t\Big[|\!\uparrow\rangle\langle\downarrow\!|\left(|1,0\rangle\langle 1,1| + |1,-1\rangle\langle 1,0|\right) + |\!\downarrow\rangle\langle\uparrow\!|\left(|1,1\rangle\langle 1,0| + |1,0\rangle\langle 1,-1|\right)\Big]$

Extracting only the evolution of the muon spins requires calculating $e^{i\mathcal{H}^{Fermi}t}\, O_\mu\, e^{-i\mathcal{H}^{Fermi}t}$ (O_μ is a muon spin operator) and doing an average over nuclear spins states. One uses the "Trace" to do this:

4 $$O_\mu(t) = \mathrm{Tr}_F\{e^{i\mathcal{H}t} O_\mu e^{-i\mathcal{H}t}\} \big/ \mathrm{Tr}_F\{1\}$$

4(a) $$= \frac{1}{4}\Big\{ O_\mu \left(1 + \cos^2\tfrac{\sqrt3}{2}t + \tfrac13\sin^2\tfrac{\sqrt3}{2}t\right)$$ Part of σ_x, σ_y & σ_z

Part of σ_z
$$|\uparrow\rangle\langle\uparrow| O_\mu |\uparrow\rangle\langle\uparrow| + \left(\cos^2\tfrac{\sqrt3}{2}t + \tfrac13\sin^2\tfrac{\sqrt3}{2}t\right)|\downarrow\rangle\langle\downarrow| O_\mu |\downarrow\rangle\langle\downarrow|$$
$$+ |\downarrow\rangle\langle\downarrow| O_\mu |\downarrow\rangle\langle\downarrow| + \left(\cos^2\tfrac{\sqrt3}{2}t + \tfrac13\sin^2\tfrac{\sqrt3}{2}t\right)|\uparrow\rangle\langle\uparrow| O_\mu |\uparrow\rangle\langle\uparrow|$$

Parts of σ_x & σ_y
$$+ |\uparrow\rangle\langle\uparrow| O_\mu |\downarrow\rangle\langle\downarrow|\, e^{-\frac{3i}{2}t}\left(\cos\tfrac{\sqrt3}{2}t - \tfrac{i}{\sqrt3}\sin\tfrac{\sqrt3}{2}t\right)$$
$$+ |\downarrow\rangle\langle\downarrow| O_\mu |\uparrow\rangle\langle\uparrow|\, e^{\frac{3i}{2}t}\left(\cos\tfrac{\sqrt3}{2}t + \tfrac{i}{\sqrt3}\sin\tfrac{\sqrt3}{2}t\right)$$
$$+ |\downarrow\rangle\langle\downarrow| O_\mu |\uparrow\rangle\langle\uparrow|\, e^{-\frac{3i}{2}t}\left(\cos\tfrac{\sqrt3}{2}t - \tfrac{i}{\sqrt3}\sin\tfrac{\sqrt3}{2}t\right)$$
$$+ |\uparrow\rangle\langle\uparrow| O_\mu |\downarrow\rangle\langle\downarrow|\, e^{\frac{3i}{2}t}\left(\cos\tfrac{\sqrt3}{2}t + \tfrac{i}{\sqrt3}\sin\tfrac{\sqrt3}{2}t\right)$$

Part of σ_z
$$+ \tfrac23\sin^2\tfrac{\sqrt3}{2}t\; 2\left(|\uparrow\rangle\langle\downarrow| O_\mu |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\uparrow| O_\mu |\uparrow\rangle\langle\downarrow|\right)\Big\}$$

Let $O_\mu = O_\alpha \sigma_\alpha^\mu$, note $A_\alpha B_\alpha \equiv \sum_\alpha A_\alpha B_\alpha$

4(b) $$= \frac14\Big\{(O_\alpha\sigma_\alpha)\left(1+\cos^2\tfrac{\sqrt3}{2}t + \tfrac13\sin^2\tfrac{\sqrt3}{2}t\right)$$
$$+ O_z\sigma_z\left(1+\cos^2\tfrac{\sqrt3}{2}t + \left(\tfrac13-\tfrac43\right)\sin^2\tfrac{\sqrt3}{2}t\right)$$
$$+ (O_x\sigma_x + O_y\sigma_y)\, 2\Big(\cos\tfrac32 t\,\cos\tfrac{\sqrt3}{2}t$$
$$- \tfrac{1}{\sqrt3}\sin\tfrac32 t\,\sin\tfrac{\sqrt3}{2}t\Big)\Big\}$$

(note: see green comments above)

4c) $$= O_z\sigma_z\left[\frac{1}{2}\left(1+\cos^2\frac{\sqrt{3}}{2}t - \frac{1}{3}\sin^2\frac{\sqrt{3}}{2}t\right)\right]$$

$$+\left(O_x\sigma_x + O_y\sigma_y\right)\left[\left(\frac{1}{4}\left(1+\cos^2\frac{\sqrt{3}}{2}t + \frac{1}{3}\sin^2\frac{\sqrt{3}}{2}t\right)\right.\right.$$

$$\left. + \frac{1}{4}\left\{\left(1+\frac{1}{\sqrt{3}}\right)\cos\left(\frac{3+\sqrt{3}}{2}\right)t + \left(1-\frac{1}{\sqrt{3}}\right)\cos\left(\frac{3-\sqrt{3}}{2}\right)t\right.\right]$$

4d) $$= O_z\sigma_z\left[2/3 + 1/3\cos\sqrt{3}t\right]$$

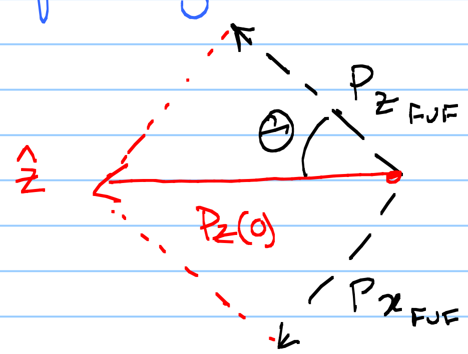
$$+\left(O_x\sigma_x + O_y\sigma_y\right)\frac{1}{4}\left\{\frac{5}{3} + \frac{1}{3}\cos\sqrt{3}t + \right.$$

$$\left.\left(1+\frac{1}{\sqrt{3}}\right)\cos\left(\frac{3+\sqrt{3}}{2}\right)t + \left(1-\frac{1}{\sqrt{3}}\right)\cos\left(\frac{3-\sqrt{3}}{2}\right)t\right\}$$

This then is the generalized evolution operator of the muon spin polarizat^{n} in a reference frame with the μ^+ spin quantized along the FuF bond.

In a general reference frame with the FuF frame at an angle Θ from the original direction of μ^+ polarization one can decompose the μ^+

5) pol^{n} as a component $\parallel$ and $\perp$ to the FuF bond axis & therefore write



6 $\langle P_z(t)\rangle = P_z(0)\left[\langle\cos^2\theta\rangle P_{z_{FuF}}(t) + \langle\sin^2\theta\rangle P_{x_{FuF}}(t)\right]$

For a distribution of $\cos\theta$ & $\sin\theta$ that has cubic (or higher, i.e. spherically symmetry)

$\langle\cos^2\theta\rangle = 1/3$ & $\langle\sin^2\theta\rangle = 2/3$

7 $\Rightarrow \langle P_z(t)\rangle = \frac{1}{3}\left[2/3 + 1/3\cos\sqrt{3}t\right]$ from the z_{FuF} component of $P_z(0)$

$+\frac{2}{3}\left[5/12 + \frac{1}{12}\cos\sqrt{3}t\right]$ from the x_{FuF} & y_{FuF} components of $P_z(0)$

$+\frac{2}{3}\cdot\frac{1}{4}\left\{\left(1+\frac{1}{\sqrt{3}}\right)\cos\left(\frac{3+\sqrt{3}}{2}\right)t + \left(1-\frac{1}{\sqrt{3}}\right)\cos\left(\frac{3-\sqrt{3}}{2}\right)t\right\}$

$= \frac{1}{6}\left\{3 + \cos\sqrt{3}t + \left(1+\frac{1}{\sqrt{3}}\right)\cos\left(\frac{3+\sqrt{3}}{2}\right)t + \left(1-\frac{1}{\sqrt{3}}\right)\cos\left(\frac{3-\sqrt{3}}{2}\right)t\right\}$

which is the "Isotropic" FuF polarization.

Note the F-F dipolar coupling was not included, but the calculat^n can be easily modified, since all the eigenstates i.e. the FuF spin 1 & spin 0 states, are still good!